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# $f$ -Factors in bipartite $(mf)$ -graphs

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## Abstract

Katerinis and Tsikopoulos (Graphs. Combin. 12 (1996) 327) give sufficient conditions for a regular bipartite graph to have a perfect matching excluding a set of edges. In this paper, we give a necessary and sufficient condition for a bipartite graph to have an  $f$ -factor containing a set of edges and excluding another set of edges and discuss some applications of this condition. In particular, we obtain some sufficient conditions related to connectivity and edge-connectivity for a bipartite  $(mf)$ -graph to have an  $f$ -factor with special properties and generalize the results in (Graphs. Combin. 12 (1996) 327). The results in this paper are in some sense best possible. © 2003 Elsevier B.V. All rights reserved.

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## 1. Preliminary and results

All graphs considered are finite undirected graphs which may have multiple edges but no loops. Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The degree of a vertex  $x$  is denoted by  $d_G(x)$ . Set  $\delta(G) = \min\{d_G(x) : x \in V(G)\}$ . The connectivity and edge-connectivity of  $G$  are denoted by  $\kappa(G)$  and  $\lambda(G)$ , respectively. Let  $f$  be a positive integer-valued function defined on  $V(G)$ . Then an  $f$ -factor of  $G$  is a spanning subgraph  $H$  of  $G$  satisfying  $d_H(x) = f(x)$  for each  $x \in V(H)$ . In particular,  $G$  is called an  $f$ -graph if  $G$  itself is an  $f$ -factor. If  $f$  is the constant function taking the value  $k$ , then an  $f$ -factor is said to be a  $k$ -factor. An  $f$ -factorization  $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$  of a graph  $G$  is a partition of  $E(G)$  into edge-disjoint  $f$ -factors  $F_1, F_2, \dots, F_m$ . Let

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$S \subseteq V(G)$  and  $D \subseteq E(G)$ . Then the subgraphs induced by  $S$  and  $D$  are denoted by  $G[S]$  and  $G[D]$ , respectively. The graph  $G[V(G) \setminus S]$  is also denoted by  $G - S$ . The graph obtained from  $G$  by adding a set of edges  $F$  is denoted by  $G + F$ . Let  $S, T \subseteq V(G)$ . Then  $E_G(S, T)$  denotes the set of edges of  $G$  having one end-vertex in  $S$  and the other in  $T$ . Write  $e_G(S, T) = |E_G(S, T)|$ . A subset  $S$  of  $V(G)$  is called a *vertex cover* of  $G$  if every edge of  $G$  has at least one end-vertex in  $S$ . The minimum cardinality of vertex cover of  $G$  is denoted by  $c(G)$ . Let  $C$  and  $D$  be two disjoint subsets of  $E(G)$ . If  $H$  is an  $f$ -factor of  $G$  such that  $C \subseteq E(H)$  and  $E(H) \cap D = \emptyset$ , then we say that  $H$  contains  $C$  and excludes  $D$ . We simply write  $g(S) = \sum_{x \in S} g(x)$  for any function  $g$  and define  $g(\emptyset) = 0$ .

A graph denoted by  $G = (X, Y, E(G))$  is a bipartite graph with bipartition  $\{X, Y\}$  and edge set  $E(G)$ . Let  $f$  be a positive integer-valued function defined on  $V(G)$ . For any  $S \subseteq X$  and  $T \subseteq Y$ , set

$$\gamma_G(S, T, f) = f(S) - f(T) + e_G(X \setminus S, T).$$

Necessary and sufficient conditions for the existence of  $f$ -factors in bipartite graphs were first given by Ore [6].

**Theorem A.** *Let  $G = (X, Y, E(G))$  be a bipartite graph and  $f$  be a positive integer-valued function defined on  $V(G)$ . Then  $G$  has an  $f$ -factor iff  $f(X) = f(Y)$  and for any  $S \subseteq X$  and  $T \subseteq Y$ , there holds*

$$\gamma_G(S, T, f) = f(S) - f(T) + e_G(X \setminus S, T) \geq 0.$$

Katerinis and Tsikopoulos [5] gave sufficient conditions for an  $m$ -regular bipartite graph to have a perfect matching excluding a set of edges.

**Theorem B.** *Let  $G = (X, Y, E(G))$  be an  $m$ -regular graph and let  $D$  be a set of edges of  $G$  such that  $|D| = m + r$ . If  $\kappa(G) \geq c(G[D]) + r + 1$  and  $\delta(G - D) \geq 1$ , then the graph  $G - D$  has a perfect matching.*

Plesnik [2] discussed the relationship between connectivity of regular graphs and the existence of 1-factors. He prove that every  $(r - 1)$ -edge connected  $r$ -regular graph of even order has a 1-factor containing a given edge and another 1-factor excluding  $r - 1$  given edges where  $r$  is an integer. Bollobás et al. [3] studied the regular factors in regular graphs. Kano [4] give sufficient conditions for a graph to have  $f$ -factors with special properties. The interested reader may find many relevant results about factors in [2,7]. In this paper, we give a necessary and sufficient condition for a bipartite graph to have an  $f$ -factor containing a set of edges and excluding another set of edges which is different from Kano's results in [4] and discuss some applications of this condition. In particular, we obtain some sufficient conditions related to connectivity and edge-connectivity for a bipartite  $(mf)$ -graph to have an  $f$ -factor with special properties and generalize Katerinis's and Tsikopoulos's results [5]. Since an  $(mf)$ -graph has an  $f$ -factorization [2], the deletion of any set of at most  $m - 1$  edges leaves an  $f$ -factor. We examine what happens if we delete more than  $m - 1$  edges. In the following we

always assume that  $f$  is a positive integer-valued function defined on  $V(G)$  and replace  $e_G(T, X \setminus S)$  with  $d_{G-S}(T)$ . For any subset  $F$  of  $E(G)$  let

$$\alpha_F = |F \cap E_G(S, Y \setminus T)|, \quad \beta_F = |F \cap E_G(T, X \setminus S)|.$$

Our main theorems are as follows.

**Theorem 1.** Let  $G = (X, Y, E(G))$  be a bipartite graph with  $f(X) = f(Y)$  and let  $C$  and  $D$  be two disjoint subsets of  $E(G)$ . Then  $G$  has an  $f$ -factor containing  $C$  and excluding  $D$  iff for all  $S \subseteq X$  and  $T \subseteq Y$ , there holds

$$\gamma_G(S, T, f) = f(S) - f(T) + d_{G-S}(T) \geq \alpha_C + \beta_D.$$

**Theorem 2.** Let  $G = (X, Y, E(G))$  be a bipartite  $(mf)$ -graph with edge-connectivity  $\lambda(G) \geq n$  and  $m \geq 2$ . Then  $G$  has an  $f$ -factor containing any given edge and excluding any given  $\lfloor (n-1)/2 \rfloor$  edges.

**Theorem 3.** Let  $G = (X, Y, E(G))$  be a bipartite  $(mf)$ -graph and  $D$  be a set of edges such that  $|D| = m + r$ ,  $r \geq 0$ . If (a)  $\kappa(G) \geq c(G[D]) + r + k$  where  $k = \min\{f(x) : x \in V(G)\}$  and (b)  $d_{G-D}(x) \geq f(x)$  for all  $x \in V(G)$ , then  $G$  has an  $f$ -factor excluding  $D$ .

**Theorem 4.** Let  $G = (X, Y, E(G))$  be a bipartite  $(mk)$ -regular graph and let  $D$  be a set of edges such that  $|D| = mk - k + 1 + r$ ,  $r \geq 0$ . If (a')  $\kappa(G) \geq c(G[D]) + r + k$  and (b')  $\delta(G - D) \geq k$ , then  $G$  has an  $k$ -factor excluding  $D$ .

Set  $n = 2m$  in Theorem 2. Then we get the following result.

**Corollary 5.** Let  $G = (X, Y, E(G))$  be a  $(2m)$ -edge connected bipartite  $(mf)$ -graph. Then  $G$  has an  $f$ -factor containing any given edge and excluding any given  $m - 1$  edges.

Note that for any graph  $G$ ,  $\kappa(G) \leq \delta(G)$ . From Theorem 4 the following Corollary can be immediately obtained.

**Corollary 6.** Let  $G$  be a bipartite  $(mk)$ -graph and let  $D$  be a subset of  $E(G)$  with  $|D| = c(G[D]) + r$ . If  $\kappa(G) \geq c(G[D]) + r + k$ , then  $G$  has a  $k$ -factor excluding  $D$ .

Theorem 1 plays a crucial role in the proof of our other theorems. The results in Theorem 2 and Corollary 5 are useful in orthogonal factorizations and combinatorial designs [1,7]. Theorems 3 and 4 and Corollary 6 are the generalizations of Theorem B. In [5] Katerinis and Tsikopoulos described a family of graphs who show that the conditions of Theorem B is best possible in some sense. When  $f(x) = k$  for all  $x \in V(G)$ , Theorem 4 is stronger than Theorem 3 if  $k \geq 2$  and Corollary 6 is stronger than Theorem 3 if  $c(G[D]) > m$ . In the next section, we will prove the above results and construct examples to show that the assertions of Theorem 2 (Corollary 5), Theorem 4 and Corollary 6 are best possible and Theorem 3 is best possible when  $c(G[D]) = m$  and  $k = 1$ . Unfortunately, we do not know whether the assertion of Theorem 3 is best possible when  $m > c(G[D])$  or  $k \geq 2$ .

## 2. Proof of the theorems

At first let us prove Theorem 1 which give a necessary and sufficient condition for a bipartite graph to admit an  $f$ -factor containing  $C$  and excluding  $D$ .

**Proof of Theorem 1.** First, let us show that  $G$  has an  $f$ -factor  $H$  such that  $E(H) \cap D = \emptyset$  iff  $\gamma_G \geq \beta_D$ . Indeed, set  $G' = G - D$ . Then the desired  $f$ -factor exists iff  $G'$  has an  $f$ -factor iff, recall Theorem A, for any  $S \subseteq X$  and  $T \subseteq Y$ ,

$$\gamma_{G'}(S, T, f) = f(S) - f(T) + d_{G'-S}(T) \geq 0.$$

Since  $d_{G'-S}(T) = d_{G-S}(T) - \beta_D$ , it follows that  $\gamma_{G'}(S, T, f) = \gamma_G(S, T, f) - \beta_D$ . Therefore,  $\gamma_{G'}(S, T, f) \geq 0$  iff  $\gamma_G(S, T, f) \geq \beta_D$ . Second, let us prove that  $G$  has an  $f$ -factor containing  $C$  iff  $\gamma_G(S, T, f) \geq \alpha_C$ . For this purpose, set  $f'(x) = d_G(x) - f(x)$ . Then the desired  $f$ -factor exists iff  $G$  has an  $f'$ -factor excluding  $C$ . By the first statement, this is equivalent to that  $\gamma_G(S, T, f') \geq \beta_C$ . Note that  $\gamma_G(S, T, f') = f'(S) - f'(T) + d_{G-S}(T) = d_G(S) - f(S) - d_G(T) + f(T) + d_{G-S}(T) = f(T) - f(S) + d_{G-T}(S) = \gamma_G(T, S, f)$ . Hence,  $G$  has an  $f$ -factor containing  $C$  iff

$$\gamma_G(T, S, f) = f(T) - f(S) + d_{G-T}(S) \geq \beta_C,$$

that is,

$$\gamma_G(S, T, f) = f(S) - f(T) + d_{G-S}(T) \geq \alpha_C$$

as desired.  $G$  has an  $f$ -factor  $H$  containing  $C$  and excluding  $D$  iff  $G'$ , defined as before, has an  $f$ -factor  $H$  containing  $C$ . By the preceding statement, this is equivalent to that  $\gamma_{G'}(S, T, f) \geq \alpha_C$ . Note that  $\gamma_{G'}(S, T, f) = \gamma_G(S, T, f) - \beta_D$ . Hence,  $G$  has an  $f$ -factor  $H$  containing  $C$  and excluding  $D$  iff  $\gamma_G(S, T, f) \geq \alpha_C + \beta_D$ .  $\square$

To prove other theorems the following lemma is necessary.

**Lemma 1.** Let  $G = (X, Y, E(G))$  be a bipartite  $(mf)$ -graph. Then for any  $S \subseteq X$  and  $T \subseteq Y$ , there holds

$$\gamma_G = \gamma_G(S, T, f) = \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S).$$

**Proof.** According to the definition of  $\gamma_G(S, T, f)$ , we have

$$\begin{aligned} \gamma_G(S, T, f) &= f(S) - f(T) + d_{G-S}(T) \\ &= d_G(T) - e_G(S, T) - f(T) + f(S) \\ &= \left( \frac{1}{m} d_G(T) - f(T) \right) + \left( f(S) - \frac{1}{m} d_G(S) \right) \\ &\quad + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \end{aligned}$$

$$\begin{aligned}
&= (f(T) - f(T)) + (f(S) - f(S)) + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\
&= \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S).
\end{aligned}$$

completing the proof.  $\square$

**Proof of Theorem 2.** We only prove the case that  $n$  is odd. The case that  $n$  is even can be justified likewise. Since  $G$  is an  $mf$ -graph, we have  $f(X) = f(Y)$ . Let  $e_0, e_1, e_2, \dots, e_{(n-1)/2}$  be any  $(n+1)/2$  edges of  $G$ . Set  $C = \{e_0\}$  and  $D = \{e_1, e_2, \dots, e_{(n-1)/2}\}$ . We prove that  $G$  has an  $f$ -factor containing  $C$  and excluding  $D$ . By Theorem 1, it suffices to show that for any  $S \subseteq X$  and  $T \subseteq Y$ ,

$$\gamma_G(S, T, f) = f(S) - f(T) + d_{G-S}(T) \geq \alpha_C + \beta_D.$$

By the definitions of  $\alpha_C$  and  $\beta_D$ , clearly,  $\alpha_C \leq 1$ ,  $\beta_D \leq (n-1)/2$  and  $\alpha_C + \beta_D \leq (n+1)/2$ . Let  $k = \min\{f(x) : x \in V(G)\}$ . Then  $n \leq \lambda(G) \leq \delta(G) \leq mk$ . Clearly,  $\gamma_G(\emptyset, \emptyset, f) = 0 = \alpha_C + \beta_D$ . If  $S = \emptyset$  and  $T \neq \emptyset$ , then  $\alpha_C = 0$  and  $\gamma_G(\emptyset, T, f) = d_G(T) - f(T) = (m-1)f(T) \geq (m-1)k \geq \frac{mk}{2} \geq \frac{n}{2} > \alpha_C + \beta_D$ . If  $S = X$ , then  $\beta_D = 0$  and  $\gamma_G(X, T, f) = f(X) - f(T) = f(Y) - f(T) = f(Y \setminus T) \geq \alpha_C + \beta_D$ . Now we assume that  $S \neq \emptyset$  and  $S \neq X$ . When  $T = \emptyset$ , we have  $\beta_D = 0$  and  $\gamma_G(S, \emptyset, f) = f(S) \geq \alpha_C + \beta_D$ . When  $T = Y$ ,  $\gamma_G(S, Y, f) = f(S) - f(Y) + d_{G-S}(Y) = f(S) - f(X) + mf(X \setminus S) = (m-1)f(X \setminus S) \geq (m-1)k \geq \alpha_C + \beta_D$ .

Now we consider that  $\emptyset \neq S \subset X$  and  $\emptyset \neq T \subset Y$ . Since  $\lambda(G) \geq n$ , we have  $d_{G-S}(T) + d_{G-T}(S) \geq n$ . Note that  $d_{G-S}(T) \geq \beta_D$ . By Lemma 1,

$$\begin{aligned}
\gamma_G(S, T, f) &= \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\
&\geq \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} (n - d_{G-S}(T)) \\
&\geq \frac{m-2}{m} d_{G-S}(T) + \frac{n}{m} \\
&\geq \beta_D - \frac{n-2\beta_D}{m} \geq \beta_D + \frac{n-n+1}{m}.
\end{aligned}$$

Since  $\gamma_G(S, T, f)$  is an integer, we obtain

$$\gamma_G(S, T, f) \geq \beta_D + 1 \geq \alpha_C + \beta_D.$$

as desired.  $\square$

**Remark 1.** The assertion of Theorem 2 is best possible in the following sense: There is an  $(n-1)$ -edge connected bipartite  $(mf)$ -graph which has no any  $f$ -factors containing a give edge and excluding given  $\lfloor (n-1)/2 \rfloor$  edges. For example, let  $G = (X, Y, E(G))$  be a bipartite graph with

$$X = \{x_i : 1 \leq i \leq mk\} \cup \{x'_i : 1 \leq i \leq mk\},$$

$$Y = \{y_i : 1 \leq i \leq mk\} \cup \{y'_i : 1 \leq i \leq mk\}$$

and

$$E(G) = \left( \{x_i y_j, x'_i y'_j: 1 \leq i, j \leq mk\} \setminus \left\{ x_i y_i, x'_i y'_i: 1 \leq i \leq \frac{n-1}{2} \right\} \right) \\ \cup \left\{ x_i y'_i, x'_i y_i: 1 \leq i \leq \frac{n-1}{2} \right\}$$

where  $k$  is a constant and  $n \leq mk$  is odd. It is easy to see that  $\lambda(G) = n - 1$ . Let  $f(x) = k$  for every  $x \in V(G)$  and let  $C = \{x_1 y'_1\}$  and  $D = \{x'_i y_i: 1 \leq i \leq (n-1)/2\}$ . For  $S = \{x_i: 1 \leq i \leq mk\}$  and  $T = \{y_i: 1 \leq i \leq mk\}$  we get

$$\gamma_G(S, T, f) = \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) = \frac{n-1}{2} < \alpha_C + \beta_D.$$

By Theorem 1,  $G$  has no  $f$ -factors containing  $C$  and excluding  $D$ .

When  $n$  is even, replacing  $(n-1)/2$  with  $(n-2)/2$  we can construct the desired example.

**Proof of Theorem 3.** Suppose that  $G$  does not have an  $f$ -factor excluding  $D$ . By Theorem 1, there are  $S \subseteq X$  and  $T \subseteq Y$  such that

$$\gamma_G(S, T, f) = f(S) - f(T) + d_{G-S}(T) < \beta_D. \quad (*)$$

We have the following claims.

**Claim 1.**  $f(S) \geq f(T) - k$ .

Otherwise,  $f(S) \leq f(T) - k - 1$ . Since  $G$  itself is an  $(mf)$ -factor, by Theorem A,

$$\gamma_G(S, T, mf) = mf(S) - mf(T) + d_{G-S}(T) \geq 0,$$

which implies that

$$d_{G-S}(T) \geq mf(T) - mf(S) \geq mk + m.$$

Note that  $c(G[D]) + r + k \leq \delta(G) \leq mk$ , that is,  $r \leq (m-1)k - 1$ . Therefore, by Lemma 1

$$\gamma_G(S, T, f) \geq \frac{m-1}{m} d_{G-S}(T) \geq (m-1)(k+1) \geq m+r \geq \beta_D,$$

a contradiction.

**Claim 2.**  $\emptyset \neq S \neq X$  and  $\emptyset \neq T \neq Y$ .

Note that when  $T = \emptyset$ ,  $\beta_D = 0$ ; when  $|T| = 1$ , by condition (b) of Theorem 3,  $\beta_D \leq |D| \leq (m-1)k$ ; when  $|T| \geq 2$ ,  $(m-1)f(T) \geq (m-1)k + m - 1 \geq m + r$ . Since  $G$  is an  $(mf)$ -graph, by Theorem A,  $f(X) = f(Y)$ . If  $S = \emptyset$ , then

$$\gamma_G(\emptyset, T, f) = -f(T) + d_G(T) = (m-1)f(T) \geq \beta_D.$$

If  $S = X$ , then  $\beta_D = 0$ . We have

$$\gamma_G(X, T, f) = f(X) - f(T) = f(Y) - f(T) = f(Y \setminus T) \geq \beta_D.$$

Similarly, We have  $\gamma_G(S, \emptyset, f) = f(S) \geq 0 = \beta_D$  and  $\gamma_G(S, Y, f) = f(S, Y, f) = f(S) - f(Y) + d_{G-S}(Y) = f(S) - f(X) + mf(X \setminus S) = (m-1)f(X \setminus S) \geq \beta_D$ . In all cases we get a contradiction with (\*).

**Claim 3.**  $d_{G'-S}(T) \leq k-1$  where  $G' = G - D$ .

By Claim 1,  $f(S) - f(T) \geq -k$ . If  $d_{G'-S}(T) \geq k$ , then  $\gamma_G(S, T, f) = f(S) - f(T) + d_{G-S}(T) = f(S) - f(T) + d_{G'-S}(T) + \beta_D \geq -k + k + \beta_D = \beta_D$ . This contradicts (\*).

**Claim 4.**  $d_{G-T}(S) \leq r$ .

From (\*) we may assume that  $\gamma_G(S, T, f) = \beta_D - n$ ,  $n \geq 1$ . Then by Lemma 1,

$$\begin{aligned} \beta_D - n &= \gamma_G(S, T, f) = \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\ &\geq \frac{m-1}{m} \beta_D + \frac{1}{m} d_{G-T}(S) \\ &\geq \beta_D + \frac{d_{G-T}(S) - \beta_D}{m} \end{aligned}$$

We have

$$d_{G-T}(S) \leq \beta_D - mn \leq m + r - mn \leq r.$$

We shall complete the proof of the theorem by deriving vertex cutsets of  $G$  which violate condition (a). For this purpose let  $D_T = D \cap E(T, X \setminus S)$  and

$$Q = \{x: x \in X \setminus S, \text{ there is an edge } xy \in D_T\}$$

and let  $K$  be an edge cover of minimum cardinality of the graph  $G[D_T]$ . Obviously  $D_T \subseteq E(G[Q \cup T])$ . Hence  $K \subseteq Q \cup T$ . In particular,  $|K| = c(G[D_T]) \leq c(G[Q \cup T]) \leq |Q \cup T| - 1$  since for every loopless graph  $G$  with  $p$  vertices we have  $c(G) \leq p - 1$ . Therefore  $K \subset Q \cup T$ . Let  $v \in (Q \cup T) \setminus K$ . Two cases are considered.

Case 1.  $v \in Q$ . Let

$$L = \{y: y \in Y \setminus T, \text{ there is an edge } xy \in E_G(S, Y \setminus T)\}$$

and let

$$J = \{y: y \in T, \text{ there is an edge } yx \in E_{G'}(T, X \setminus S)\}.$$

From Claims 4 and 3 we have  $|L| \leq d_{G-T}(S) \leq r$  and  $|J| \leq d_{G'-S}(T) \leq k-1$ . Hence,

$$|K \cup L \cup J| \leq |K| + |L| + |J| \leq c(G[D]) + r + k - 1.$$

It is easy to see that  $K \cup L \cup J$  disconnects  $v$  from  $S$ . Recalling Claim 2 we get a vertex cutset  $K \cup L \cup J$  which has cardinality less than  $\kappa(G)$ . This contradicts the condition (a) in the theorem.

Case 2.  $v \in T$ . Let

$$L = \{x: x \in S, \text{ there is an edge } xy \in E_G(S, Y \setminus T)\}$$

and

$$J = \{x: x \in X \setminus S, \text{ there is an edge } xy \in E_{G'}(T, X \setminus S)\}.$$

From Claims 4 and 3 we have  $|L| \leq d_{G-T}(S) \leq r$  and  $|J| \leq d_{G'-S}(T) \leq k-1$ . Hence,

$$|K \cup L \cup J| \leq |K| + |L| + |J| \leq c(G[D]) + r + k - 1.$$

Since  $K \cup L \cup J$  disconnects  $v$  from  $Y \setminus T$ , it is a vertex cutset of  $G$  which contradicts the condition (a), completing the proof.

**Proof of Theorem 4.** Set  $f(x) = k$  for every  $x \in V(G)$ . By Theorem 1, to prove the theorem it suffices to show that for any  $S \subseteq X$  and  $T \subseteq Y$ ,

$$\gamma_G(S, T, f) = k|S| - k|T| + d_{G-S}(T) \geq \beta_D.$$

If  $|S| \geq |T|$ , then  $\gamma_G(S, T, f) \geq d_{G-S}(T) \geq \beta_D$ . If  $|S| \leq |T| - 2$ , since  $G$  is an  $mk$ -regular graph, we have

$$mk|S| - d_{G-T}(S) = mk|T| - d_{G-S}(T).$$

Therefore,  $d_{G-S}(T) \geq mk|T| - mk|S| \geq 2mk$ . Note that  $mk \geq \kappa(G) \geq c(G[D]) + r + k$ . We have  $r \leq (m-1)k - 1$ . By Lemma 1,

$$\gamma_G(S, T, f) \geq \frac{m-1}{m} d_{G-S}(T) \geq 2(m-1)k \geq (m-1)k + r + 1 \geq \beta_D.$$

Now we assume that  $|S| = |T| - 1$ . Clearly,  $T \neq \emptyset$ . Since  $|X| = |Y|$ ,  $S \neq X$ . When  $S = \emptyset$ , we have  $|T| = 1$  and  $\gamma_G(\emptyset, T, f) = -k|T| + d_G(T) = (m-1)k \geq \beta_D$ . For  $\emptyset \neq S \subset X$ , and  $|S| = |T| - 1$

$$\gamma_G(S, T, f) = k|S| - k|T| + d_{G-S}(T) \geq -k + d_{G-S}(T).$$

If  $d_{G-S}(T) \geq \beta_D + k$ , then  $\gamma_G(S, T, f) \geq \beta_D$ . Hence,  $d_{G-S}(T) \leq \beta_D + k - 1$ . We have

$$d_{G'-S}(T) = d_{G-S}(T) - \beta_D \leq k - 1,$$

where  $G' = G - D$ . Thus Claim 3 in Theorem 3 holds. Since  $d_{G-S}(T) = mk|T| - mk|S| + d_{G-T}(S) \geq mk + d_{G-T}(S)$ , by Lemma 1,

$$\begin{aligned} \gamma_G(S, T, f) &= \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\ &\geq \frac{m-1}{m} (mk + d_{G-T}(S)) + \frac{1}{m} d_{G-T}(S) \\ &\geq (m-1)k + d_{G-T}(S). \end{aligned}$$

If  $d_{G-T}(S) \geq r + 1$ , then  $\gamma_G(S, T, f) \geq (m-1)k + r + 1 \geq \beta_D$ . Therefore,

$$d_{G-T}(S) \leq r.$$

Claim 4 in Theorem 3 holds. Repeating the argument after Claim 4 in Theorem 3 we can derive a vertex cutset of  $G$  with cardinality  $c(G[D]) + r + k - 1$ , which contradicts the condition (a'). The theorem is proved.  $\square$



**Remark 2.** The assertions of Theorem 4 and Corollary 6 are best possible in the following sense: Even the connectivity of  $G$  decreases, by just one, the graph  $G$  may not have a  $k$ -factor excluding  $D$ . When  $f(x)=k$  for all  $x \in V(G)$  and  $k=1$  ( $c(G[D])=m$ ), Theorem 4 (Corollary 6) is equivalent to Theorem 3. Hence, in this case Theorem 3 is best possible.

First we consider  $r=0$  and construct a graph  $G_0$  showing the sharpness of condition in Corollary 6. We denote the complete bipartite graph with bipartition  $(X, Y)$  such that  $|X|=p$  and  $|Y|=q$  by  $K_{p,q}$ . Set  $H_1=(X_1, Y_1, E(H_1))=K_{mk-1, mk}$ ,  $H_2=(X_2, Y_2, E(H_2))=K_{mk, mk} - \{x_{2j}y_{2j}: 1 \leq j \leq mk\}$  and  $H_3=(X_3, Y_3, E(H_3))=H_{mk, mk-1}$  where

$$X_1 = \{x_{1,j}: 1 \leq j \leq mk-1\}, \quad X_i = \{x_{i,j}: 1 \leq j \leq mk\}, \quad i = 2, 3,$$

$$Y_i = \{y_{i,j}: 1 \leq j \leq mk\}, \quad i = 1, 2, \quad \text{and} \quad Y_3 = \{y_{3,j}: 1 \leq j \leq mk-1\}.$$

Let  $G_0 = H_1 \cup H_2 \cup H_3 + F_1 + F_2$  where

$$F_1 = \{x_{2,j}y_{1,j}: 1 \leq j \leq mk\} \quad \text{and} \quad F_2 = \{x_{3,j}y_{2,j}: 1 \leq j \leq mk\}.$$

It is easy to see that  $G_0$  is an  $(mk)$ -connected bipartite  $(mk)$ -regular graph. Let  $D$  be a subset of  $F_1$  such that  $|D| = mk - k + 1$ . Then  $c(G_0[D]) = mk - k + 1$ . We have  $|D| = c(G_0[D]) + r, r=0$ ,  $\delta(G_0 - D) \geq k$  and  $\kappa(G_0) = mk = c(G_0[D]) + r + k - 1$ . Choosing  $S = X_1$  and  $T = Y_1$  we have  $d_{G_0-S}(T) = mk, d_{G_0-T}(S) = 0$  and  $\beta_D = |D| = mk - k + 1$ . Hence by Lemma 1,

$$\gamma_{G_0}(S, T, k) = \frac{m-1}{m} d_{G_0-S}(T) = (m-1)k < \beta_D.$$

By Theorem 1,  $G_0$  does not have a  $k$ -factor excluding  $D$ .

Now we consider  $r > 0$  and construct  $G_r$  from  $G_0$ . Set  $G_r = G_0 - E_1 + E_2$  where

$$E_1 = \{x_{2,mk}y_{2,j}, x_{2,j}y_{1,j}: 1 \leq j \leq r\}$$

and

$$E_2 = \{x_{2,j}y_{2,j}, x_{2,mk}y_{1,j}: 1 \leq j \leq r\}.$$

It is not difficult to verify that  $G_r$  is a bipartite  $(mk)$ -regular graph with  $\kappa(G_r) = mk$ . Taking

$$D = \{x_{2,j}y_{1,j}: r+k \leq j \leq mk\} \cup \{x_{2,mk}y_{1,j}: 1 \leq j \leq r\}$$

we have  $c(G_r[D]) = mk - r - k + 1$ ,  $|D| = mk - k + 1 = c(G_r[D]) + r$ ,  $\delta(G_r - D) \geq k$  and  $\kappa(G_r) = mk = c(G_r[D]) + r + k - 1$ . Let  $S = X_1$  and  $T = Y_1$ . Since

$$\gamma_{G_r}(S, T, k) = \frac{m-1}{m} d_{G_r-S}(T) = (m-1)k < mk - k + 1 = \beta_D$$

by Theorem 1,  $G_r$  has no  $k$ -factors excluding  $D$ .

Now we show that the conditions of Theorem 4 are sharp. Condition  $(b')$  is obviously necessary. We construct  $H_r$  from  $G_r$ . Set  $H_r = G_r - E_3 + E_4$  where

$$E_3 = \{x_{2,mk-1}y_{2,j}, x_{1,j}y_{1,j}: 1 \leq j \leq r\}$$

and

$$E_4 = \{x_{1,j}y_{2,j}, x_{2,mk-1}y_{1,j} : 1 \leq j \leq r\}.$$

It is not difficult to verify that  $H_r$  is a bipartite  $(mk)$ -regular graph with  $\kappa(H_r) = mk$ . Let

$$D = \{x_{2,j}y_{1,j} : r+k \leq j \leq mk\} \cup \{x_{2,mk}y_{1,j}, x_{2,mk-1}y_{1,j} : 1 \leq j \leq r\}.$$

Then  $c(H_r[D]) = mk - r - k + 1$ ,  $|D| = mk - k + 1 = c(H_r[D]) + r$ ,  $\delta(H_r - D) \geq k$  and  $\kappa(H_r) = mk = c(H_r[D]) + r + k - 1$ . Let  $S = X_1$  and  $T = Y_1$ . We have

$$\begin{aligned} \gamma_{H_r}(S, T, k) &= \frac{m-1}{m} d_{H_r-S}(T) + \frac{1}{m} d_{H_r-T}(S) \\ &= (m-1)k + \frac{r}{m} < mk - k + r + 1 = \beta_D, \end{aligned}$$

recalling Theorem 1 we have known that  $H_r$  has no  $k$ -factor excluding  $D$ . Condition  $(a')$  is also necessary.

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